The Bézoutian and the $\mathbb{A}^1$-degree

degree in classical topology:

$$f : S^n \longrightarrow S^n \quad \text{← n-sphere}$$

$$\rightsquigarrow \quad \begin{array}{ccc} f_* : H_n(S^n) & \longrightarrow & H_n(S^n) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} & & \mathbb{Z} \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ 1 & \longmapsto & \deg f \end{array}$$

$$\begin{array}{cccc} \deg : & [S^n, S^n] & \xrightarrow{\cong} & \mathbb{Z} \\ & \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ & f & \longmapsto & \deg f \end{array}$$

Let $q \in S^n$ be a regular value and $f^{-1}(q) = \{p_1, \dots, p_m\} \ni p$

local degree

$$\deg_p f = \deg\left(U/\partial U \xrightarrow{\bar{f}} V/\partial V \right)$$

U = small ball around p $\qquad f^{-1}(q) \cap U = \{p\}$

V = — " — q

$$\deg_p f = \begin{cases} +1 & \text{locally orientation preserving} \\ -1 & \text{— " — reversing} \end{cases}$$

$= \operatorname{sign}(\det \operatorname{Jac} f(p))$

Poincaré-Hopf thm

$V \xrightarrow{\sigma} X$ real oriented rank r vector bundle

n

compact oriented n-mfld

Let $\sigma : X \to V$ be a general section

Then

$$\deg e(V) = \sum_{p \in \sigma^{-1}(0)} \deg_p \sigma$$

Euler class

Example: $V = \operatorname{Sym}^3 S^\vee \to Gr(2,4) = X$

$k = \mathbb{R}$

fibers

$[\ell]$

lines in $\mathbb{P}^3$

$\operatorname{Sym}^3 S^\vee_{[\ell]}$ = degree 3 poly's on ℓ

Then # zeros of a general section σ ~~=~~ = # lines on a cubic surface

$$\overset{\shortparallel}{} $$

$$\deg e(V) = \sum_{\substack{\text{lines on} \\ \text{a smooth cubic surface}}} \operatorname{sign}(\text{line}) = 3$$

The Grothendieck-Witt ring of k

k field, char $k \neq 2$

$GW(k) :=$ group completion of semiring of isometry classes of non-degenerate quadratic forms$/k$

$\oplus$ direct sum = Addition

$$\left.\begin{array}{l} q_1 : V_1 \to k \\ q_2 : V_2 \to k \end{array}\right\} \to \begin{array}{l} q_1 \oplus q_2 : V_1 \oplus V_2 \to k \\ q_1 \otimes q_2 \end{array}$$

Any quadratic form over k can be diagonalized:

$$q(x_1, \dots, x_n) = a_1 x_1^2 + a_2 x_2^2 + \dots + a_n x_n^2$$

$$a_i \in k^\times$$

$\rightsquigarrow$ generators for $GW(k)$

$$\langle a \rangle = [a x^2] \qquad a \in k^\times / (k^\times)^2$$

relations: 1) $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle$

$a, b, a+b \neq 0$

2) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

Ex 1) $GW(\mathbb{C}) \cong \mathbb{Z}$ as a group

2) $GW(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$

$$\mathbb{R}^\times / (\mathbb{R}^\times)^2 = \{\pm 1\}$$

The $\mathbb{A}^1$-degree (Morel)

= degree in $\mathbb{A}^1$-homotopy theory

homotopy theory of smooth alg varieties $/k$

n-sphere in $\mathbb{A}^1$-homotopy theory

$$\mathbb{P}^n_k / \mathbb{P}^{n-1}_k$$

$$\deg^{\mathbb{A}^1} : [\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \longrightarrow GW(k)$$

$\mathbb{A}^1$ htpy classes

Example (lines on a cubic surface)

Kass-Wickelgren

$$\deg e^{\mathbb{A}^1}(\mathrm{Sym}^3 S^\vee \to Gr(2,4))$$

[e]

$$= \sum_{\substack{\text{lines on} \\ \text{a cubic surface}}} \mathrm{Type}(\text{line}) = 15\langle 1 \rangle + 12 \langle -1 \rangle$$

$\in GW(k)$

Local $\mathbb{A}^1$-degree (Kass-Wickelgren)

$$f : \mathbb{P}^n_k / \mathbb{P}^{n-1}_k \to \mathbb{P}^n_k / \mathbb{P}^{n-1}_k$$

y rational pt

Then

$$\deg^{\mathbb{A}^1} f = \sum_{x \in f^{-1}(y)} \deg_x^{\mathbb{A}^1} f$$

Def $(\deg_x^{\mathbb{A}^1} f)$

U: Zariski nbhd of x

V: — " — y

$$\deg \left(\mathbb{P}^n_k / \mathbb{P}^{n-1}_k \to \mathbb{P}^n_k / \mathbb{P}^n_k - \{x\} \cong U / U \setminus \{x\} \xrightarrow{\bar{f}} V / V \setminus \{y\} \cong \mathbb{P}^n_k / \mathbb{P}^n_k - \{y\} \cong \mathbb{P}^n_k / \mathbb{P}^{n-1}_k \right)$$

$$=: \deg_x^{\mathbb{A}^1} f$$

Prop (Kass- Wickelgren)

$f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ $\quad$ x is an isolated zero

and $\det \mathrm{Jac}\, f(x) \neq 0$

and assume

$k(x)/k$ separable

Then

$$\deg_x^{\mathbb{A}^1} f = \mathrm{Tr}_{k(x)/k} \langle \det \mathrm{Jac}\, f(x) \rangle$$

$$\underbrace{V \xrightarrow{\text{quadr form}} k(x) \xrightarrow{\mathrm{Tr}_{k(x)/k}} k}_{\text{quadr form } /k}$$

Prop: (Kass-Wickelgren, Brazelton-Burklund-McKean-Montoro-Opie)

Assume $k(x)/k$ separable

Then

$$\deg_x^{\mathbb{A}^1} f = \mathrm{Tr}_{k(x)/k} (\text{EKL-form})$$

EKL-form:

$(f_1, \dots, f_n) = f : \mathbb{A}^n_k \to \mathbb{A}^n_k$

x isolated zero

$$Q_x := \frac{k[x_1, \dots, x_n]_x}{(f_1, \dots, f_n)} \quad \text{finite } k\text{-alg}$$

$E :=$ image of $\det b_{ij}$ in Q_x

where

$$b_{ij} \in k[x_1, \dots, x_n]$$

st $$\sum_j b_{ij} \cdot x_j = f_i$$

Choose basis $a_1, \dots, a_m$ of Q_x

$$a_m = E$$

Let $\Phi : Q_x \to k$ k-linear

st $a_i \mapsto 0$ $i < m$

$a_m = E \mapsto 1$

Then EKL-form

$=$ quadr form with Gram matrix

$$\Phi(a_i \cdot a_j)$$

Ex: $f : \mathbb{A}^1 \to \mathbb{A}^1$

$$x \mapsto x^n$$

$E = x^{n-1}$ basis $1, x, \dots, x^{n-1}$ for $Q_0 = \dfrac{k[x]_0}{(x^n)}$

Gram matrix

$$\begin{array}{c|c}
 & \begin{matrix} 1 & x & \cdots & x^{n-1} \end{matrix} \\
\begin{matrix} 1 \\ x \\ x^2 \\ \vdots \\ \\ x^{n-1} \end{matrix} & \begin{pmatrix} 0 & & \cdots & 0 & 1 \\ 0 & & & 1 & 0 \\ \vdots & & \iddots & & \\ & \iddots & & & 0 \\ 0 & 1 & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}
\end{array}$$

$$\rightsquigarrow \begin{cases} \frac{n}{2} \cdot \left(\langle 1 \rangle + \langle -1 \rangle \right) & n \text{ even} \\ \frac{n-1}{2} \left(\langle 1 \rangle + \langle -1 \rangle \right) + \langle 1 \rangle & n \text{ odd} \end{cases}$$

Bézoutian

$f = (f_0 : f_1) : \mathbb{P}^1_k \to \mathbb{P}^1_k$

$$x := \frac{x_1}{x_0} \qquad y = \frac{y_1}{y_0}$$

$$\text{Béz} := \frac{f_1(x) \cdot f_0(y) - f_1(y) \cdot f_0(x)}{x - y}$$

$$= \sum B_{ij} \, x^i y^j \qquad B_{ij} \in K$$

B_{ij} is the Gram matrix of a quadratic form.

Thm (Cazanave)

$$\deg^{\mathbb{A}^1} f = [(B_{ij})] \in GW(k)$$

$\exists$ multivariate Bézoutian:

$f = (f_1, \dots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$

with only isolated zeros

$$D_{ij} := \frac{f_i(Y_1, \dots, Y_{j-1}, X_j, \dots, X_n) - f_i(Y_1, \dots, Y_j, X_{j+1}, \dots X_n)}{X_j - Y_j}$$

$\in k[X_1, \dots, X_n, Y_1, \dots, Y_n]$

$$Q := \frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_n)}$$ finite k-algebra

Béz := image of $\det D_{ij}$

in $Q \otimes_k Q$

$$\| $$

$$\frac{k[X_1, \dots, X_n, Y_1, \dots Y_n]}{(f(X), f(Y))}$$

Choose a k-vector space basis

$a_1, \dots, a_m$ of Q

$\rightsquigarrow$ $\{ a_i(X) \otimes a_j(Y) \}$ is a basis

of $Q \otimes_k Q$

Can write

$$\text{Béz} = \sum B_{ij} \, a_i(X) \otimes a_j(Y)$$

$$B_{ij} \in k$$

Thm (Brazelton - McKean - P.)

(B_{ij}) is the Gram matrix

of $\deg^{\mathbb{A}^1} f = \sum_{\substack{\text{zeros} \\ x}} \deg_x^{\mathbb{A}^1} f$

Also works to compute $\deg_x^{\mathbb{A}^1} f$

$\rightsquigarrow$ localize Q at x

This also works for non-separable field ext $k(x)/k$

Ex: p odd prime

$k = \mathbb{F}_p(t)$

$$f = (f_1, f_2) : \mathbb{A}^2_k \longrightarrow \mathbb{A}^2_k$$
$$\parallel$$
$$(x_1^p - t,\ x_1 x_2)$$

$$D_{11} = \frac{(x_1^p - t) - (y_1^p - t)}{x_1 - y_1} \qquad D_{12} = 0$$

$$D_{21} = \frac{x_1 x_2 - y_1 x_2}{x_1 - y_1} = x_2 \qquad D_{22} = y_1$$

$$\text{Béz} = \det \begin{pmatrix} \frac{x_1^p - y_1^p}{x_1 - y_1} & 0 \\ x_2 & y_1 \end{pmatrix}$$

$$= y_1 x_1^{p-1} + y_1^2 x_1^{p-2} + \dots + y_1^{p-1} x_1 + y_1^p$$

$= t$

$\rightsquigarrow$ Gram matrix

$$\begin{array}{cc} & \begin{matrix} 1 & y_1 & \cdots & y_1^{p-1} & \end{matrix} \\ \begin{matrix} 1 \\ x_1 \\ \vdots \\ x_1^{p-1} \end{matrix} & \begin{pmatrix} t & 0 & \cdots & 0 & 0 \\ 0 & & & & 1 \\ \vdots & 0 & & \cdot^{\cdot^{\cdot}} & \\ 0 & 1 & & & 0 \end{pmatrix} \end{array}$$

$$\rightsquigarrow \langle t \rangle + \frac{p-1}{2} \cdot \left(\langle 1 \rangle + \langle -1 \rangle \right)$$

"Nisnevich coordinates"

$x \in X \leftarrow$ smooth

Can always find:

U nbhd of x

$+ \quad \psi : U \xrightarrow{\text{étale}} \mathbb{A}^n$

st ψ is iso on $k(x)$